THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, April 27, 1901. Thirty-four persons were present at the two sessions, including the following thirty-one members of the Society :

Mr. Joseph Allen, Dr. J. E. Clarke, Professor F. N. Cole, Dr. W. S. Dennett, Professor T. S. Fiske, Mr. A. S. Gale, Dr. G. B. Germann, Mr. F. A. Giffen, Dr. W. A. Granville, Miss Ida Griffiths, Professor James Harkness, Dr. Edward Kasner, Mr. C. J. Keyser, Professor Pomeroy Ladue, Dr. G. H. Ling, Dr. Emory McClintock, Dr. James Maclay, Dr. G. A. Miller, Professor Frank Morley, Professor W. F. Osgood, Miss Ida Schottenfels, Professor C. A. Scott, Dr. Virgil Snyder, Dr. H. F. Stecker, Professor J. H. Tanner, Professor H. D. Thompson, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor L. A. Wait, Miss E. C. Williams, Professor R. S. Woodward.

Vice-President Thomas S. Fiske occupied the chair. The Council announced the election of the following persons to membership in the Society : Mr. C. W. McG. Black, Harvard University, Cambridge, Mass.; Dr. S. E. Slocum, University of Cincinnati, Cincinnati, Ohio. Two applications for membership were received.

To relieve the increasing burden of administration, the office of Assistant Secretary was created, and filled by the appointment of Dr. Edward Kasner, to serve until February, 1902.

The library of the Society, which at present consists mainly of some five hundred unbound volumes of journals received as exchanges, is about to be deposited in the library of Columbia University, under an agreement by which the University undertakes to bind, catalogue, and care for the books now on hand and all future additions, and to make them easily accessible to the members of the Society. Arrangements will be made by which the books may be temporarily loaned to members living at a distance. The library is to be kept as a separate collection, duplicating as far as may be the general University library, and aiming to become as complete as possible in itself. The title to the books remains in the Society, which reserves the right to withdraw them under agreed conditions.

The following papers were read at this meeting:

(1) Dr. W. A. GRANVILLE: "Invariants of some m-gons under certain projective Lie groups in the plane."

(2) Dr. EDWARD KASNER: "The algebraic potential surfaces."

(3) Professor F. MORLEY: "On the real foci of algebraic curves."

(4) Mr. GEORGE PEIRCE: "A curious approximate construction for π."

(5) Professor E. W. HYDE: "On a surface of the sixth order, which is touched by all screws belonging to a three-conditioned system."

(6) Professor L. E. DICKSON: "The hyper-orthogonal groups."

(7) Professor E. W. BROWN: "On least action and minimal surfaces."

(8) Professor W. H. METZLER: "On certain aggregates of determinant minors."

(9) Professor E. B. VAN VLECK: "On the convergence of continued fractions with complex elements; supplementary note."

(10) Professor W. F. OSGOOD: "On a fundamental property of a minimum in the calculus of variations and the proof of a theorem of Weierstrass."

(11) Professor E. O. LOVETT: "The geometry of quadrics."

(12) Professor E. O. LOVETT: "The differential geometry of n-dimensional space."

(13) Dr. G. A. MILLER: "On the groups generated by two operators."

(14) Dr. EDWARD KASNER: "The relations between the angles of any number of lines in n-space."

(15) Dr. L. P. EISENHART: "Isothermal conjugate systems of lines on surfaces."

(16) Dr. E. J. WILCZYNSKI: "Geometry of a simultaneous system of two linear homogeneous differential equations of the second order."

(17) Dr. H. F. BLICHFELDT: "A new determination of the primitive continuous groups in two variables."

Mr. Peirce's paper was communicated to the Society by Professor E. W. Brown. In the absence of the authors, the papers of Mr. Peirce, Professor Hyde, Professor Dickson, Professor Brown, Professor Metzler, Professor Lovett, Dr. Eisenhart, Dr. Wilczynski, and Dr. Blichfeldt were read by title. The papers of Mr. Peirce and Dr. Miller will appear in the BULLETIN. Abstracts of the other papers are given below.

Dr. Granville finds the invariant functions of certain configurations under the substitutions of some of the important subgroups of the general projective group in the plane. Problems of this sort are among the most important and interesting in Lie's theory of continuous groups. First an outline is given of the theory followed in solving the problems. Then the invariant functions of a quadrangle under the substitutions of the group

$$\boxed{xp,\ yp,\ xq,\ yq,\ x^2p + xyq,\ xyp + y^2q}$$

are found. The results do not agree with those found by Professor Lovett in the *Annals of Mathematics*, volume 12, where the same problem is proposed. The remaining four problems solved are of the same nature.

The surfaces considered by Dr. Kasner in his first paper are obtained by equating the rational integral potential functions $\varphi(x, y, z)$ to zero; their analytic property is expressed by the equation

$$\Delta\varphi \equiv \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0.$$

It is shown that the surfaces may be defined geometrically by their relation to the imaginary circle at infinity: the surfaces are apolar to this circle when the latter is considered as a degenerate surface of the second class. The potential surfaces of the second order are rectangular hyperboloids, *i. e.*, their asymptotic cones contain triples of mutually orthogonal generators. The polar quadrics of any potential surface are rectangular hyperboloids; conversely, a surface is potential when all its polar quadrics are rectangular hyperboloids. Potential cylinders and cones are considered in detail. The results are finally extended to n-dimensional space.

The point of Professor Morley's note was that, given any line equation of an algebraic curve

$$f(\xi_1, \cdots, \xi_n) = 0,$$

where ξ_r is the distance from a fixed point a_r to the moving line, the real foci are given by writing in the terms of highest order $x - a_r$ for ξ_r, where $x - a_r$ is the stroke from a fixed point to a focus. Some applications of this simple rule were given.

Any three screws determine a system of screws in space consisting of the "totality of the generators of one system of a skew conicoid which changes in accordance with the variation of a parameter involved in its equation." Professor Hyde assumes a particular arrangement of the determining screws, viz.; they are mutually perpendicular, and have a common point. The envelope of the before-mentioned skew conicoid is found, which will be touched by all the screws of the system. It is a surface of the sixth order whose constants are functions of the differences of the pitches of the determining screws, and is shown to be reducible to a comparatively simple form in which only two of these differences appear. The peculiarities of the surface are discussed, and three diagrams given, illustrating the different forms which it may assume.

In a paper published in the *Mathematische Annalen*, volume 52, pp. 561–581, Professor Dickson showed that the structures of the groups defined by invariants

$$\sum_{i=1}^{m} \lambda_i \xi_i^r \qquad (r>2)$$

in any Galois field depend upon the structure of the group with the invariant

$$\sum_{i=1}^{m} \xi_i \xi_2^{p^s+1}$$

in the $GF[p^{2s}]$. The latter group is called the hyperorthogonal group H, since it includes the orthogonal group as a subgroup. It may be represented as a transitive substitution group on N_c symbols

$$\lambda_1\xi_1 + \cdots + \lambda_m\xi_m$$

where $\lambda_1, \cdots, \lambda_m$ are any marks, not all zero, satisfying

$$\lambda_1^{p^s+1} + \cdots + \lambda_m^{p^s+1} = c = \text{constant}.$$

For any $c \neq 0$, we have

$$N_c = p^{s(2m-1)} - (-1)^m p^{s(m-1)};$$

while, for $c = 0$, N_c has the value

$$\omega_m \equiv [p^{ms} - (-1)^m][p^{(m-1)s} - (-1)^{m-1}].$$

The substitutions of H which multiply every index by the

same constant form an invariant subgroup of H. The quotient-group $HO(m, p^{2s})$ is a simple group except when $m = 2$, $p^s = 2$; $m = 2$, $p^s = 3$; $m = 3$, $p^s = 2$. Except when $m = 2$, $p^s = 2$, the quotient group may be represented as a transitive substitution group on $N_c \div (p^s + 1)$ letters for $c \neq 0$, as well as upon $\omega_m \div (p^{2s} - 1)$ letters for $c = 0$. The latter number is the smaller except for $p^s = 2$, m even, $m > 2$, when the former number is $\frac{1}{3}(2^{2m-1} - 2^{m-1})$. For $m = 2$, the smaller number is $p^s + 1$, a result in accord with the known isomorphism of $HO(2, p^{2s})$ with the linear fractional group in the $GF[p^s]$.

The paper makes a detailed study of the characteristic equations of hyperorthogonal substitutions. If x be one root, then is x^{-p^s} also a root. Inversely, given an equation having this property, we can determine a hyperorthogonal substitution having it as its characteristic equation. Such an equation is never irreducible in the $GF[p^{2s}]$ if $m \neq 3$; it never decompose into linear factors and a single irreducible factor of degree $\neq 3$. If $f_t(x)$ be one irreducible factor, there exists a complementary irreducible factor

$$x^t \{ f_t(x^{-p^{ts}}) \}^{p^s}$$

of equal degree t. Thus, for $m = 4$, if $x^2 - \alpha x + \beta$ be an irreducible factor, then is also

$$x^2 - x\alpha^{p^s}\beta^{-1} + \beta^{-1},$$

where β necessarily belongs to the included $GF[p^s]$. The greater part of the paper is devoted to the reduction of hyperorthogonal substitutions to canonical forms within the group and to their distribution into complete sets of conjugate substitutions. The results are exhaustive for the case $m = 3$. For example, when $p^s = 3$, $m = 3$, the group H is a simple group of order 6048, containing exactly 14 non-conjugate types of substitutions whose periods are 1, 2, 3, 4, 6, 7, 8, 12.

Professor Brown showed that the problem of the motion of a particle in a plane is reducible to the discovery of a certain cylindrical surface whose area between two generating lines is a minimum. A similar result is also shown for the case of a particle whose motion is referred to axes moving in the plane with uniform velocity.

In 1888 (*Proceedings of the Royal Society of Edinburgh*, pp. 99–105) Dr. T. Muir showed that a linear relation exists

between certain minors of a centro-symmetric determinant similar to Kronecker's relation between the minors of an axi-symmetric determinant; and in 1900 (same journal, pp. 142–154) he gave two theorems connecting the minors of any determinant, the first of which reduces to Kronecker's relation on imposing the conditions for axi-symmetry, and the second of which reduces to his 1888 relation on imposing the conditions for centro-symmetry. Professor Metzler extends these two theorems given by Dr. Muir so as to include his as special cases and gives a whole series of types of linear relations between the minors of a centro-symmetric determinant, Muir's being one type of the series. The paper also gives the number of relations of each type.

Professor Van Vleck presented a note supplementary to his paper read at the February meeting. The note will be embodied in the paper, which will appear in the *Transactions*.

Professor Osgood's paper, which will appear in the *Transactions*, is in abstract as follows: The integral

$$I = \int_{x_0}^{x_1} F(x, y, y')dx$$

or, in the parametric form,

$$I = \int_{t_0}^{t_1} F(x, y, x', y')dt$$

is said to be made a minimum by the functions $x = \varphi(t)$, $y = \psi(t)$ corresponding to the curve C, if its value J corresponding to this curve is less than its value I corresponding to any other curve $\bar{C}$ lying in a neighborhood T of C and satisfying the boundary conditions of the problem. Weierstrass has given a sufficient condition for a minimum. The question, however, presents itself: Assuming that Weierstrass's sufficient condition is fulfilled, so that no curve $\bar{C}$, distinct from C, of the class of curves admitted to consideration will give the integral I so small a value as J, may there not still be a set of curves belonging to this class, $\bar{C}_1, \bar{C}_2, \cdots$ which do not cluster about C as their limit, and which have the property that, if $I_1, I_2, \cdots$ denote respectively the values of the integral I formed for these curves,

$$\lim_{n=\infty} I_n = J?$$

This question is answered in the negative. It is shown that, if $\mathfrak{T}$ denotes an arbitrarily small neighborhood of C lying wholly within T, the lower limit of I formed for all the curves C of the class in question that do not lie wholly within $\mathfrak{T}$ is greater than J by a positive quantity ε, varying for different regions $\mathfrak{T}$, but constant for any given region; so that

$$I \geqq J + \varepsilon, \qquad \text{when } \bar{C} \text{ does not lie wholly in } \mathfrak{T}.$$

By means of this result it is possible to extend the class of curves $\bar{C}$ with reference to which the integral I possesses the minimum property and thus establish the existence of a more general minimum. Making the generalized definition of the length of a curve his point of departure, Weierstrass inscribed in an arbitrary curve $\bar{C}$ that satisfies the boundary conditions, is continuous, and lies within T, a polygon and formed the sum

$$S = \sum_{i=0}^{n-1} F\left(x_i, y_i, \frac{\Delta x_i}{\Delta t_i}, \frac{\Delta y_i}{\Delta t_i}\right) \Delta t_i = \sum_{i=0}^{n-1} F(x_i, y_i, \Delta x_i, \Delta y_i).$$

If S converges toward one and the same limit, no matter how the vertices of the polygon be chosen, provided merely that all its sides converge toward 0, this limit is taken as the generalized limit $\mathfrak{J}$ and the class of curves admitted to consideration is enlarged so as to include $\bar{C}$. Weierstrass states the theorem that $\mathfrak{J} > J$ if $\bar{C}$ is distinct from C. This theorem is proven in the present paper and it is furthermore shown that

$$\mathfrak{J} \geqq J + \varepsilon \qquad \text{when } \bar{C} \text{ does not lie wholly in } \mathfrak{T}.$$

The theory of the group of euclidean motions yields the notions of ordinary n-dimensional geometry. In his first paper Professor Lovett shows that it can also be made to yield the elements of this geometry. The group is extended relative to the coefficients of the equation of the general quadric. An application of Euler's theorem facilitates the construction of the invariants. In particular it appears that the squares of the semi-axes of the quadric

$$\sum_{1}^{n}\sum a_{ij}x_ix_j + 2\sum_{j=1}^{n} a_{j,\,n+1}x_j + a_{n+1,\,n+1} = 0$$

are given by the equation

$$I_{n+1}^{n} + \sum_{1}^{n-1} I_n^i I_i I_{n+1} x^i + I_n x^n = 0,$$

where

$$I_k = \sum_1^n {}_{i_1 i_2 \cdots i_k} |a_{i_1 i_1}, a_{i_2 i_2}, \cdots, a_{i_k i_k}|.$$

Every space of k dimensions within the n-dimensional space has its own differential geometry. To determine its elements, analogous to those of Gauss for the differential geometry of surfaces in ordinary space, it is only necessary to construct the nth extension of the group of euclidean motions relative to k independent parameters which are assumed to be invariant under the transformations of the group.

Professor Lovett's second paper effects these extensions for the values 1 and $n-1$ of k, and constructs the corresponding differential invariants. In particular, it appears that the following relations are the generalizations for four dimensional space of the three well-known relations of Gauss, Mainardi, and Codazzi among the Gaussian fundamental quantities of the first and second order :

$$2 \cdot 12_{12} - 11_{22} - 22_{11} = \varphi_1, \quad 2 \cdot 23_{23} - 22_{33} - 33_{22} = \varphi_2,$$

$$2 \cdot 31_{31} - 33_{11} - 11_{33} = \varphi_3,$$

$$12_{23} - 13_{23} = \varphi_4, \quad 13_{22} - 12_{23} = \varphi_5, \quad 13_{12} - 12_{13} = \varphi_6,$$

$$12_2 - 22_1 = \psi_1, \quad 23_3 - 33_2 = \psi_2, \quad 22_3 - 23_2 = \psi_3,$$

$$11_2 - 12_1 = \psi_4, \quad 12_3 - 23_1 = \psi_5, \quad 11_3 - 31_1 = \psi_6,$$

$$12_3 - 31_2 = \psi_7, \quad 33_1 - 31_3 = \psi_8,$$

where

$$\ddot{v}_{kl} = E_{\psi_{u_k u_l}}, \qquad \ddot{v}_k = F_{\psi_{u_k}},$$

$$E_{ij} = \sum_1^n {}_k \frac{\partial x_k}{\partial u_i} \frac{\partial x_k}{\partial u_j}, \qquad F_{ij} = \sum_1^n {}_k X_k \frac{\partial^2 x_k}{\partial u_i \partial u_j},$$

$$X_i |E_{11}, E_{22}, \cdots, E_{n-1\,n-1}|^{\frac{1}{2}} = M_i,$$

in which M_i is the determinant formed by suppressing the ith column of the matrix

$$||x_{1_{u_1}}, x_{2_{u_2}}, \cdots, x_{n_{u_n}}||,$$

and the φ's and ψ's are functions of the first and second order differential invariants whose forms are easily constructed. The theory of this group gives the whole theory of k-parametric displacements in a space of n-dimensions, as a future note will show.

Dr. Kasner's second paper shows that, in space of n dimensions, the angles of any number of lines less than $n+1$ are all independent; while the $\frac{1}{2}n(n+1)$ angles of $n+1$ lines $L_1, L_2, \cdots, L_{n+1}$ are connected by the single relation

$$\begin{vmatrix} 1 & (1,2) & \cdots & (1,n+1) \\ (2,1) & 1 & \cdots & (2,n+1) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ (n+1,1) & (n+1,2) & \cdots & 1 \end{vmatrix} = 0,$$

where (i,j) denotes the cosine of the angle L_i, L_j. The relations for any number of lines are of the same type. For ordinary space this gives the known relation between the sides and diagonals of a spherical quadrilateral, and the relation between the dihedral angles of a tetrahedron.

Dr. Eisenhart's paper is, in abstract, as follows: An extension of the term isothermal-conjugate system of lines, as given by Bianchi to those systems on a surface of positive curvature for which as parametric curves the second fundamental form reduces to $\lambda(du^2+dv^2)$, is made to those conjugate systems on surfaces of negative curvature for which this form reduces to $\lambda(du^2-dv^2)$, where λ is a function of u and v. Results are found for these systems very similar to those given by Bianchi for the former case. A general study is made of these lines on both kinds of surfaces; the determination of these systems on surfaces of positive curvature is effected by methods similar to those used by Bianchi in the study of isothermal orthogonal systems. A geometrical interpretation of isothermal-conjugate systems leads to the theorem that the surfaces obtained by reciprocal radii vectores from surfaces whose lines of curvature are isothermal-conjugate are of the same kind. The greater part of the paper is devoted to a study of surfaces whose lines of curvature are isothermal-conjugate, and it is shown that the general problem of determining such surfaces depends upon the integration of a differential equation of the fourth order, very similar to the equation found by Darboux in his study of isothermic surfaces. It is noted that the surfaces whose lines of curvature are at the same time isothermal and isothermal-conjugate have an isothermal spherical representation, and from this is deduced the theorem that the sphere and minimal surfaces are the only surfaces of constant mean curvature whose lines of curvature are isothermal-conjugate; also that the quadric surfaces have isothermal-conjugate lines of curvature.

It is shown that the plane, sphere, cyclides of Dupin, and surfaces of revolution are the only surfaces which together with their parallels have an isothermal conjugate system of lines of curvature; and that the surfaces of constant total curvature, the surfaces of revolution, and the surfaces whose radii satisfy two definite equations of condition (not given here) are the only Weingarten surfaces possessing the above property. Associated with a surface with isothermal-conjugate systems of lines of curvature is a second surface with isothermal lines of curvature, corresponding to the former by parallelism of tangent planes; it can be found by quadratures. In particular, when the first surface is of constant total curvature, the second is of constant mean curvature and parallel to the former. The paper closes with a discussion of surfaces with plane lines of curvature in both systems which at the same time are isothermal-conjugate. Solutions of this problem are furnished by surfaces of revolution, certain moulded surfaces, cyclides of Dupin, minimal surfaces of Bonnet and Enneper, and other surfaces whose equations are given.

In a paper recently published in the *Transactions*, Dr. Wilczynski has initiated a theory of invariants of a system of linear differential equations. In the present paper, the author considers a remarkable relation between this theory and line geometry. If (y_i) and (z_i) $(i = 1, 2, 3, 4)$ form a fundamental system of simultaneous solutions of the system of differential equations

$$(1) \qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z &= 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z &= 0, \end{aligned}$$

and y_i and z_i be interpreted as the homogeneous coördinates of two points in space, it is seen that (1) defines two curves C_y and C_z as its integral curves. The points (y_i) and (z_i) of these curves, which correspond to the same value of the independent variable x, are joined by a straight line L_{yz}, and these straight lines generate a ruled surface S, called the integrating ruled surface of system (1), which, it is shown, can never be a developable surface.

The most general transformations which convert (1) into another system of the same kind are

$$(2) \qquad x = f(\xi), \quad y = \alpha\eta + \beta\xi, \quad z = \gamma\eta + \delta\xi,$$

where f, α, β, γ, δ are arbitrary functions of ξ. It is noted that these transformations leave invariant the ruled surface

S, merely converting the two integral curves into any other two curves on this surface. Invariant relations, it is shown, give projective properties of the integrating ruled surface. The differential equation of the sixth order, satisfied by the line coördinates of a generator of the ruled surface, is set up in a normal form. Then the necessary and sufficient conditions are discussed under which the generators of the ruled surface belong to a linear complex, either general or special, to a linear congruence with distinct or coincident directrices, and finally the conditions under which they form a surface of the second order. The reduction of the system to the semi-canonical form is found to be equivalent with the geometrical problem of finding the curved asymptotic lines on the surface, and a new proof of Serret's theorem about such lines is given.

The primitive continuous group of point transformations in two variables can, by a proper choice of the variables, be transformed into *projective groups* of the plane, a result Lie obtains after determining the canonical forms of the primitive groups. This fact can, however, be established from the general properties of such groups, and it is shown in Dr. Blichfeldt's paper, which will appear in the *Transactions*, that this point of view gives rise to a new determination of these primitive groups.

EDWARD KASNER,
Assistant Secretary.

COLUMBIA UNIVERSITY.

THE APRIL MEETING OF THE CHICAGO SECTION.

A REGULAR meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago on Saturday, April 6, 1901. The following members were present:

Professor W. W. Beman, Professor Oskar Bolza, Professor D. F. Campbell, Professor E. W. Davis, Professor Thomas F. Holgate, Professor E. H. Moore, Dr. F. R. Moulton, Miss Ida M. Schottenfels, Professor H. E. Slaught, Mr. Burke Smith.

The President of the Society, Professor E. H. Moore, occupied the chair. The following papers were presented and read:

(1) Dr. F. R. MOULTON: "On the validity of the method of computing absolute perturbations."

(2) Professor GEORGE C. CHATBURN: "A theorem in arrangements."

(3) Professor E. W. DAVIS: "A definition of mathematical probability."

(4) Dr. JACOB WESTLUND: "Note on multiply perfect numbers."

(5) Professor F. MERTENS: "Zur linearen Transformation der ϑ-Reihen."

(6) Dr. J. H. MCDONALD: "The reduction of hyperelliptic integrals of genus two to elliptic integrals, by a transformation of order four."

(7) Professor L. E. DICKSON: "Representation of linear groups as transitive substitution groups."

(8) Dr. G. A. MILLER: "Determination of all the groups of order p^m which contain the abelian group of the type $(m-2, 1)$, p being any prime number."

(9) Mr. ROBERT E. MORITZ: "A generalization of the differentiation process."

(10) Mr. ROBERT E. MORITZ: "On ratients, with an extension of the ordinary calculus."

Professor Chatburn's paper and the two papers by Mr. Moritz were presented to the Society through Professor Davis, and in the absence of the authors were read by him; Professor Mertens's paper was presented through Professor Moore; in the absence of Dr. Westlund his paper was read by the Secretary. Dr. McDonald was introduced to the Section by Professor Bolza. Abstracts of the papers follow.

It is generally understood, and indeed it is usually stated in works on celestial mechanics,* that the methods used by astronomers are only approximate, or else that the convergence of the series employed is assumed. The former means to a mathematician that the functions from which the perturbations are computed do not represent the actual perturbations with rigor; the latter leaves the validity of the whole process in doubt. In Dr. Moulton's paper it is asked whether the power series in the masses used in computing the perturbations are convergent, and whether they rigorously represent these perturbations. It is answered that, under certain conditions which are fulfilled in case of the major planets, the series in the masses, in actual use by as-

* Dziobek's Planeten-Bewegung, p. 167; Tisserand's Mécanique céleste, vol. 1, pp. 190 and 193–4; Brown's Lunar theory, preface, p. vi.

tronomers, are absolutely convergent for all values of the time not greater in numerical value than a fixed limit which can be determined, and that these series represent the perturbations exactly. This limit within which the series are certainly convergent is doubtless much smaller than the true radius of convergence. The results are equally true in Laplace's method of computing the absolute perturbations of the coördinates, and in Lagrange's method of the variation of parameters.

The method of proof is an adaptation of Cauchy's calcul des limites,* which has been used with great success in the well known works of Fuchs, Frobenius, Weierstrass, Poincaré, and many others, in the theory of differential equations. The particular modification used here is the same extension as that employed by Poincaré in his memoir in the *Acta Mathematica*, volume XIII. Indeed, Poincaré's results are closely related to those attained here; but, having another object in view, his investigation does not regard the validity of the precise method used by astronomers.

The second part of the paper is devoted to a discussion of the geometrical and physical meaning of the terms of the various orders with respect to each mass separately. It is shown that there is a perfect correspondence between the terms of the power series and the successive corrections to the elliptic orbits which, from a general mathematical point of view, seem to lead most naturally toward the true orbit. The conclusions are that the methods used by astronomers are, under certain limitations, perfectly valid, and that "successive approximations" means successive approximations toward the true numerical values by means of series which represent the functions exactly.

Professor Chatburn's theorem in arrangements gives a method by which c things (say cards numbered consecutively) can be divided into groups, and then rearranged group by group, redivided and rearranged for n or more times, so that the final position of any one of the things shall be that given by the terms of an arithmetical progression in which unity is the first term, c the last term, and p the number of terms.

Professor Davis offered the following definition for mathematical probability: the direct probability that the occasion A shall lead to the event B is the value toward which tends

* *Comptes rendus*, vol. 14.

the ratio of the number of A's that are followed by B, to the total number of A's, as these recur without limit. He also stated a corresponding definition for inverse probability.

Defining a multiply perfect number as one which is an exact divisor of the sum of all its divisors, the multiplicity being the quotient, Dr. Westlund shows in his note that the only numbers of multiplicity 3 of the form $m = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3$ where p_1, p_2, p_3 are three distinct primes, are the two numbers $2^3 \cdot 3 \cdot 5$ and $2^5 \cdot 3 \cdot 7$.

Professor Mertens makes a direct and elegant determination of the twenty-fourth root of unity, fundamental in the theory of the linear transformations of the θ-series. His paper will be published in the *Transactions*.

The reduction of hyperelliptic integrals of genus two to elliptic integrals is effected by a special involution of order four containing a form which is the square of a quadratic. From the general Weierstrass-Picard theorem it is known that if there exists one reducible integral, there exists also a second having the same irrationality and reducible by a transformation of the same order. The problem treated in Dr. McDonald's paper is: Given one integral, to find the other. It is first shown that the special involution of order four may be uniquely determined by a cubic and a quadratic form. Among the covariants of this system of cubic and quadratic there exists another cubic and quadratic having with the first mutual relations analogous to, but more perfect than, the relations between a single cubic form and its cubic covariant. The involution which is determined by this second cubic and quadratic is one which will deduce a second hyperelliptic integral having the same irrationality as the first but a different numerator, which must be the one indicated by the Weierstrass-Picard theorem. The relations between the two numerators of reducible integrals and the corresponding involutions are a generalization of those found by Professor Bolza for the reduction of order three, and which also exist for the reduction of order two.

In Professor Dickson's paper, which is to be published in the *American Journal of Mathematics*, it is pointed out that the group of all linear homogeneous substitutions S on m indices with coefficients in the $GF[p^n]$ may be represented as a transitive substitution group on $p^{nm} - 1$ letters; the simple group defined by taking the substitutions S frac-

tionally may be represented as a doubly transitive substitution group on $(p^{nm}-1)/(p^n-1)$ letters. In the former case we may employ as the $p^{nm}-1$ letters the totality of functions $\lambda \equiv \lambda_1\xi_1 + \lambda_2\xi_2 + \cdots + \lambda_m\xi_m$, in which the λ_i run independently through the series of marks of the field, the case $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$ being excluded.

For a subgroup defined by an invariant $\psi(\xi_1, \cdots, \xi_m)$, the above representation leads to an intransitive substitution group. To illustrate the method of obtaining a transitive representation, consider the orthogonal group. If an orthogonal substitution replaces ξ_1 by $\omega \equiv a_1\xi_1 + \cdots + a_m\xi_m$ then $a_1^2 + \cdots + a_m^2 = 1$; inversely, for any such set of marks a_i, the orthogonal group contains a substitution replacing ξ_1 by ω. Under an orthogonal substitution the above functions λ are permuted in such a way that the totality of functions λ in which $\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_m^2$ is a constant mark c are permuted transitively. Except when $\psi S = 3$, the case $c = 0$ leads to a representation upon fewer letters than any case $c \neq 0$. For $c = 0$, we combine the p^{n-1} functions λ, in which the ratios $\lambda_1 : \lambda_2 : \cdots : \lambda_m$ are fixed, into one symbol $\xi\lambda\xi$. We obtain for $m > 2$ a transitive representation of the orthogonal quotient group. According as m is odd or even, this minimum number of letters is $(p^{n(m-1)}-1)/(p^n-1)$ or

$$(p^{nm/2} - \varepsilon)(p^{n(m/2-1)} + \varepsilon)/(p^n - 1)$$

$$[\varepsilon \equiv (-1)^{m(p^n-1)/4}]$$

For $m = 3$, the number is $p^n + 1$, in accord with the known isomorphism of the orthogonal quotient group with the linear fractional group in one variable.

These methods are then applied to the representation of every linear group defined by a quadratic invariant. The first hypoabelian group on $2m$ indices in the $GF[2^n]$ is represented as a transitive substitution group on either of the following numbers of letters:

$$(2^{nm}-1)(2^{n(m-1)}+1)/(2^n-1),\ (2^{nm}-1)2^{n(m-1)},$$

the second number being larger than the first if $n > 1$ and smaller if $n = 1$. For the case $n = 1$, the small number is $2^{2m-1} - 2^{m-1}$, in accord with Jordan's proof of the isomorphism of the first hypoabelian group with the groups studied by Steiner in dealing with problems of contact of curves of the m^{th} order The second hypoabelian group is represented as a transitive substitution group on $(2^{nm}+1)(2^{n(m-1)}-1)/(2^n-1)$ letters.

In a paper to be presented to the *Mathematische Annalen*, the method is applied to the hyperorthogonal groups with the invariant $\sum_{i=1}^{m} \xi_i \bar{\xi}_i$.

Dr. Miller's paper deals in the first place with the group of isomorphisms of the abelian group H of type $(m-2, 1)$. This study is facilitated by the use of two theorems (proved in the introduction), viz.: The only operators that transform each subgroup of an abelian group into itself are those which transform each operator of the group into the same power, and the number of invariant operators in the group of isomorphisms of an abelian group is equal to the number of natural numbers which do not exceed the highest order of operators of the group and are prime to this order.

The group of isomorphisms of H contains $p^3 - 1$ operators of order p (when p is odd) forming, with identity, the non-abelian group of order p^3. When $p = 2$ there are 15 operators of order 2 in the group of isomorphisms of H. These generate a group of order 32 containing 16 operators of order four. When $p > 3$ there are seven non-abelian groups of order p^m which contain the abelian group of type $(m-2, 1)$; when $p = 3$ there are eight such groups and the number of these groups is seventeen when $p = 2$. The paper will be published in the *Transactions*.

Mr. Moritz's first paper, which will be published in the *American Journal of Mathematics*, is an extension of the ideas presented in a paper by the same author, of which an abstract is found on page 185, volume 6, of the BULLETIN. Starting with an abstract conception of an indeterminate, he arrives at the conclusion that the process of differentiation does not occupy the unique position which it is commonly supposed to occupy, but that there are other limiting processes. The equation $\lim_{h=1} \log_h \frac{F(x^h)}{F(x)} = \frac{gy}{gx}$ defines such a process, to which the term *quotientiation* is applied, $\frac{gy}{gx}$ being the quotiential coefficient of $y = F(x)$ with respect to x. The laws of this process are examined and the various rules for its application developed. It is suggested that a quotiential in which quotientiation takes the place of differentiation in the ordinary calculus, is not only conceivable, but might be applied, though with less ease than the differential calculus, to the investigation and interpretation of natural phenomena.

A further attempt at setting up other forms fails apparently, owing to the unsymmetric nature of the process a^b. If however $a^{\log b}$ is looked upon as derived by successive multiplication of a, just as $a \times b$ is obtained by successive additions of a, the difficulty vanishes completely. In fact, using De Morgan's form of the algebraic processes in which the process of the nth order is defined by $\log^{-n}(\log^n a + \log^n b)$, its inverse by $\log^{-n}(\log^n a - \log^n b)$, where n may be negative as well as positive, and where the 0th and 1st process are respectively identical with the ordinary addition and multiplication process, the writer finds that for every pair of consecutive processes there exists a limiting process, which when applied to any function $y = F(x)$ gives rise to a function $\frac{d_n y}{d_n x} = \log^{-n}\left(\frac{d \log^n y}{d \log^n x}\right)$, where n is the lower order of the processes involved. $\frac{d_n y}{d_n x}$ is called the *ratient* in the nth process of y with respect to x. Since n may be negative as well as positive, the chain of ratients can be extended in either direction without limit, the zero ratients being the ordinary differential coefficients. It is shown how any ratient may be expressed in terms of the ratient of any higher or lower order process. Successive ratients assume the form $\frac{d_n{}^r y}{d_n x^r} = \log^{-n} \frac{d^r \log^n y}{(d \log^n x)^r}$.

Mr. Moritz's second paper is based upon the definition of De Morgan's processes, and that of ratients as developed by the author in his paper on "Generalization of the differentiation process." A formula is developed for the ratient in the nth process of two or more functions combined by the mth process, from which the differentiation formulæ for a product of functions, powers of functions, or generally of functions combined by the process of any order, positive or negative, follow among others as simple corollaries. It is then shown that the fundamental theorems regarding the differentiation of a sum or a product hold for ratients in general when sums and products are replaced respectively by operands in the nth and $(n+1)$th processes.

The number system for the nth process is examined and it is found that every De Morgan process possesses a distinctive number system whose properties are identical with those of the ordinary number system. To the imaginary i in the ordinary algebra correspond new imaginaries, to the transcendentals π and e correspond new transcendentals. In fact

the transcendentals corresponding to π and e in the nth process are algebraic numbers borrowed from the $(n-1)$th and $(n+1)$th processes respectively. The totality of numbers may thus be conceived as arranged in an infinite number of strata, the ordinary algebraic numbers forming a single stratum.

With the extension of the differentiation process, and the stratification of the number body, all the results of the ordinary calculus, for example Taylor's theorem, can be expressed in terms of any two consecutive processes.

THOMAS F. HOLGATE,
Secretary of the Section.

EVANSTON, ILL.

THE VALUE OF $\int_0^{\pi/2} (\log 2\cos\varphi)^m \varphi^n d\varphi$.

BY PROFESSOR F. MORLEY.

(Read before the American Mathematical Society, April 29, 1899.)

THE integral in question, in which m is any positive integer, and n is any even positive integer or zero, is given in effect, for the cases $m=1$, $n=0$, in Williamson's Integral Calculus, § 118 of the second edition, being taken from a paper by H. G. (presumably Harvey Goodwin, Bishop of Carlisle) in volume 3 of the *Quarterly Journal of Mathematics*. Further it is given in effect, for the case $m=2$, $n=0$, in Wolstenholme's Problems, p. 332 of the second edition. It seems worth while to show how it can be expressed in general, in terms of the constants $s_r = \sum_1^\infty 1/n^r$, which may be regarded as known.

We know that when p and q are real

$$1 + \frac{p\cdot q}{1\cdot 1} + \frac{p(p-1)q(q-1)}{1\cdot 2\cdot 1\cdot 2} + \cdots = \frac{\Gamma(1+p+q)}{\Gamma(1+p)\Gamma(1+q)},$$

when $p+q > -1$ (Forsyth, Differential equations, p. 197). But the left member is the constant term in the product of the series

$$(1+x)^p = 1 + px + \cdots$$

and

$$(1+1/x)^q = 1 + q/x + \cdots.$$

We suppose p and q positive, so that both series are absolutely convergent when $|x| = 1$. We may then multiply the series and get a Laurent series for

$$(1 + x)^p(1 + 1/x)^q.$$

Hence

$$\frac{1}{2i\pi}\int_{|x|=1}(1 + x)^p(1 + 1/x)^q\frac{dx}{x} = \frac{\Gamma(1 + p + q)}{\Gamma(1 + p)\Gamma(1 + q)}.$$

The path of integration shall be from -1 positively round the circle; the point -1 being a branch point of the integral, but not an infinity.

Writing $x = \exp 2i\varphi$, the integral becomes

$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(2\cos\varphi)^{p+q}\exp i(p - q)\varphi d\varphi.$$

Let

$$p + q = 2\alpha, \quad p - q = 2\beta;$$

then

$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\exp\{2\alpha\log(2\cos\varphi) + 2i\beta\varphi\}d\varphi = \frac{\Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha + \beta)\Gamma(1 + \alpha - \beta)}.$$

Both sides can be developed in the form

$$\Sigma\Sigma C_{m,n}\alpha^m\beta^n;$$

the left directly by the exponential theorem, giving

$$C_{m,n} = \frac{2^{m+n}}{\pi m!\ n!}\int_{-\pi/2}^{\pi/2}(\log 2\cos\varphi)^m(i\varphi)^n d\varphi,$$

while the right, obtained from the formula

$$\log\Gamma(1 + n) = -\gamma n + \sum_2^\infty(-)^r\frac{\zeta_r}{r}x^r,$$

is

$$\exp\left[\sum_2^\infty(-)^r\frac{\zeta_r}{r}\{(2\alpha)^r - (\alpha + \beta)^r - (\alpha - \beta)^r\}\right].$$

Hence the result: The integral

$$\frac{2}{\pi}\int_0^{\pi/2}(\log 2\cos\varphi)^m\varphi^n d\varphi$$

where m is any positive integer, n any positive even integer or 0, is

$$(-)^{n/2}\frac{m!\,n!}{2^{m+n}} \times \text{the coefficient of } \alpha^m\beta^n \text{ in}$$

$$\exp\left[\sum_2^\infty (-)^r \frac{s_r}{r}\{(2\alpha)^r - (\alpha+\beta)^r - (\alpha-\beta)^r\}\right].$$

This generating function is, to a few terms,

$$\exp\,[s_2(\alpha^2-\beta^2) - 2\,\alpha s_3(\alpha^3-\beta^3)$$
$$+\tfrac{1}{2}s_4(7\alpha^4 - 6\alpha^2\beta^2 - \beta^4) - 2\,\alpha s_5(3\alpha^4 - 2\alpha^2\beta^2 - \beta^4) + \,]$$

or

$$1 + s_2(\alpha^2-\beta^2) - 2s_3(\alpha^3 - \alpha\beta^2)$$
$$+\frac{s_2^{\,2}+7s_4}{2}\alpha^4 - (s_2^{\,2}+3s_4)\alpha^2\beta^2 + \frac{s_2^{\,2}-s_4}{2}\beta^4$$
$$-2(s_2s_3+3s_5)\alpha^5 + 4(s_2s_3+s_5)\alpha^3\beta^2 - 2(s_2s_3-s_5)\alpha\beta^4 + \cdots$$

Thus, in tabular form, a few values for

$$\frac{2}{\pi}\int_0^{\pi/2}(\log 2\cos\varphi)^m\varphi^n d\varphi$$

are:

$n=$	0	2	4
$m=0$*	1	$\frac{1}{2}s_2$	$\frac{3}{4}(s_2^{\,2}-s_4)$
1	0	$-\frac{1}{2}s_3$	$-\frac{3}{2}(s_2s_3-s_5)$
2	$\frac{1}{2}s_2$	$\frac{1}{4}(s_2^{\,2}+3s_4)$	
3	$-\frac{3}{2}s_3$	$-\frac{3}{2}(s_2s_3+s_5)$	
4	$\frac{3}{4}(s_2^{\,2}+7s_4)$		
5	$-\frac{15}{2}(s_2s_3+3s_5)$		

ON THE ALGEBRAIC POTENTIAL CURVES.

BY DR. EDWARD KASNER.

(Read before the American Mathematical Society, February 23, 1901.)

THE object of this paper is to derive the characteristic geometric properties of a class of curves which are of in-

* The row for which $m=0$ is of course merely a verification, leading to the known values

$$s_2 = \pi^2/6, \quad s_4 = \pi^4/90.$$

terest in connection with the theory of equations and of the potential function. Analytically, these curves are obtained by equating to zero the rational integral solutions $\varphi(x, y)$ of Laplace's equation

$$\Delta\varphi \equiv \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0,$$

or, what is equivalent, the real (or imaginary) parts of the rational integral functions of $x + iy$. Various geometric properties are given in Briot and Bouquet's Théorie des fonctions elliptiques (book IV, chapter II), but none are completely characteristic.

§ 1. *Apolarity with Respect to a Point Pair.*

A curve

$$\Phi \equiv a_x^n = 0,$$

is said to be apolar* to a conic

$$Q \equiv \Sigma P_{ik} u_i u_k \equiv u_P^2 = 0,$$

when every polar conic of the curve is circumscribed about an infinite number of triangles self-conjugate with respect to the conic Q; *i. e.*, when the bilinear covariant

$$S \equiv \Sigma P_{ik} \frac{\partial^2\Phi}{\partial x_i \partial x_k} \equiv a_P^2 a_x^{n-2} \tag{1}$$

vanishes identically.

Let the conic degenerate into a pair of points A, B,

$$A \equiv u_\alpha = 0, \quad B \equiv u_\beta = 0,$$

so that

$$Q = u_\alpha u_\beta;$$

then

$$S = a_\alpha a_\beta a_x^{n-2},$$

which is the apolar covariant of the forms

$$u_\alpha \quad \text{and} \quad u_\beta a_x^{n-1},$$

or of

$$u_\beta \quad \text{and} \quad u_\alpha a_x^{n-1}.$$

It is easy to show, however, that a point and a curve can be apolar only when the curve consists of a set of straight lines passing through the point, so that

* Reye, *Crelle*, vol. 79 (1874), p. 159. For a convenient summary of the theory of apolar relations see Schlesinger, *Math. Annalen*, vol. 22 (1883), pp. 520–523.

THEOREM I. *If a curve of the nth order is apolar to a point pair (considered as a degenerate conic), the first polar (and therefore any of the polars) of either of the points with respect to the curve consists of a set of lines passing through the other point; the converse is also true.*

Letting $n = 2$, it follows that a conic is apolar to a point pair when the two points are conjugate with respect to the conic. From the definition of apolarity we have then

THEOREM II. *If a curve is apolar to a point pair, the latter is self-conjugate with respect to all the polar conics of the curve; conversely, etc.*

Since from Theorem I both the first polar and the polar conic of either point have nodes, we have

THEOREM III. *If a curve is apolar to a point pair, both the Hessian and the Steinerian of the curve pass through the point pair; furthermore, these points correspond in the sense defined by Clebsch.**

§2. *Polar Properties of Potential Curves.*

Instead of an arbitrary point pair, consider now the pair of circular points at infinity I, J. The equation of this point pair in rectangular line coördinates may be written

$$u^2 + v^2 = 0;$$

so that, expressed in rectangular point coördinates, the covariant S of the preceding section becomes

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}.$$

The vanishing of this expression, however, denotes that the curve $\varphi = 0$ is a potential curve. Therefore,

THEOREM IV. *Any potential curve is apolar to the fundamental conic of euclidean geometry consisting of the circular points at infinity; conversely, any curve which is apolar to this fundamental conic is a potential curve.*†

From Theorem I we have then

THEOREM V. *All the polar curves of a circular point with respect to a potential curve degenerate into sets of straight lines passing through the other circular point; conversely, etc.*

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*Clebsch, "Ueber einige von Steiner behandelte Kurven," *Crelle*, vol. 64, p. 288. The converse of the above theorem is not true.

†Cf. Clifford, "On the canonical form of spherical harmonics," Works, p. 234, for a statement concerning "nodal curves" on a sphere, which appears to have some connection with the above.

A conic is a potential curve when the circular points I, J are conjugate with respect to it; this implies that the conic intersects the line at infinity in points which are harmonic with respect to I and J, *i. e.*, that the asymptotes of the conic are rectangular. From Theorem II, we have then a statement of the characteristic property of potential curves which has the advantage of dealing only with real elements, as follows:

THEOREM VI. *A curve is a potential curve when, and only when, the polar conics of all points with respect to the curve are rectangular hyperbolas.*

From Theorem III we have a property, which is however not characteristic, *i. e.*, not restricted to the potential curves:

THEOREM VII. *The Hessian and the Steinerian curves of a potential curve pass through the circular points I, J; furthermore, these points correspond in the sense defined by Clebsch.*

Since the polar conics of a polar curve are the polar conics of the original curve, we have

THEOREM VIII. *All the polar curves of a potential curve are themselves potential curves.*

§ 3. *Focal Properties of Potential Curves.*

Consider any rational integral function of the nth order in z

$$f(x+iy) \equiv \varphi(x, y) + i\psi(x, y), \tag{2}$$

together with the conjugate expression

$$\bar{f}(x-iy) \equiv \varphi(x, y) - i\psi(x, y); \tag{3}$$

the equation of the potential curve $\varphi = 0$ may be written in the form

$$f(x+iy) + \bar{f}(x-iy) = 0. \tag{4}$$

The two terms of the left hand member of this equation, equated separately to zero, represent sets of minimal lines, the first representing n lines through I, and the second n lines through J. Furthermore, equation (4) is unchanged when $f(z)$ is replaced by $f(z) + i\lambda$, where λ is an arbitrary real constant. We have then

THEOREM IX. *The linear system of curves of the nth order determined by $2n$ miminal lines (n through each of the circular points) is composed of potential curves; conversely, any potential curve may be obtained as a member of an infinite number of such linear systems.*

This theorem may be restated by using the fact that a curve of the nth class has n^2 foci, namely the intersections of the n minimal tangents of one system with the n minimal tangents of the other system, as follows:

THEOREM X. *Any curve of the nth order passing through the n^2 foci of a curve of the nth class is a potential curve; conversely, all potential curves may be obtained in this way—each potential curve passes through the foci of an infinite number of systems of confocal curves of the nth class.**

Thus for $n = 2$, we have that all the conics which pass through the two real and the two imaginary foci of a conic are rectangular hyperbolas.

The potential curves $\varphi = 0$, $\psi = 0$, obtained in the decomposition of a function of $x + iy$, may be termed *conjugate*† *potential curves*, since the functions φ and ψ are conjugate. From (2) and (3) we have

$$(5) \qquad \begin{aligned} 2\varphi &= f(x+iy) + \bar{f}(x-iy), \\ 2i\psi &= f(x-iy) - \bar{f}(x-iy); \end{aligned}$$

therefore the curves belong to a linear system of the kind considered above. Furthermore they intersect orthogonally.‡

THEOREM XI. *Conjugate potential curves of the nth order intersect orthogonally in the foci of a system of confocal curves of the nth class; conversely, two curves of the nth order which intersect orthogonally in the foci of a curve of the nth class are conjugate potential curves.*

The properties stated in Theorems IV and X being definitive for the same class of curves, it follows that these properties are equivalent. From this equivalence we may pass to a more general result relating to the apolarity of a curve and a point pair; it is necessary merely to project the circular points into an arbitrary pair of points, the potential curves transforming into curves which are apolar to this pair. Therefore, if through each of two points A, B, n straight lines are drawn, any curve of the nth order passing through the n^2 points so determined is apolar to the pair A, B; moreover this construction yields all the apolar curves. This result may be restated:

THEOREM XII. *A curve of the nth order is apolar to a pair of points, A, B when, and only when, it is possible to find upon*

* The number of parameters in such a confocal system is $\frac{1}{2}n(n-1)$; so that the number of curves from which any potential curve may be derived as a focal curve is $\infty^{\frac{1}{2}(n^2-n+2)}$.

† The term conjugate, of course, here refers to the properties of the functions φ, ψ, and is not synonymous with the term apolar.

‡ Briot et Bouquet, p. 223.

*the curve n^2 points lying by n's upon n lines through A and at the same time upon n lines through B.**

§ 4. *The Asymptotes.*

Briot and Bouquet prove† that the n asymptotes of a potential curve of the nth order are real, concurrent, and disposed symmetrically about their common point, the angle between consecutive asymptotes being $2\pi/n$. This property, however, imposes only $2n-3$ independent conditions, while the number imposed by the equation ‡ $\triangle\varphi \equiv 0$ is $\frac{1}{2}n(n-1)$; so that in general (*i. e.*, if $n > 3$) the above relation between the asymptotes is not peculiar to the potential curves, and Briot and Bouquet's theorem cannot be converted. As to the case $n = 2$ §, it has been shown in § 2 that the potential conics are the rectangular hyperbolas, so that the relation between the asymptotes is characteristic. The same is true in the case $n = 3$, as may be shown by taking a coördinate system with its origin at the point of concurrence of the asymptotes and its axis of abscissas coinciding with one of the asymptotes, and verifying the condition $\triangle\varphi = 0$. Therefore, *the potential cubics may be defined as those cubics which have three real concurrent asymptotes intersecting at angles of 120°.*

In all cases the *point of concurrence O of the asymptotes is a center* of the curve, *i. e.*, if any line is drawn through O, the sum of the distances measured from O of the points of intersection lying on one side of O is the same as the corresponding sum for the points on the other side. This follows from the fact that when the origin of coördinates is taken at O, all the terms of order $n-1$ disappear. ||

From the potential curves we may pass by projection to the curves which are apolar to any point pair A, B. The asymptotes of the potential curve are transformed into a set of concurrent lines tangent to the new curve at the points $P_1, P_2, \cdots, P_n$ where the line joining A, B cuts the curve. From the equality of the angles between consecutive asymptotes, the anharmonic ratios

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* When one such set of n^2 points exists there is necessarily an infinite number of sets.

† L. c., p. 227.

‡ The number of parameters in the potential curve of the nth order is $2n$, so that of all the curves of the nth order which pass through $2n$ assigned points one and only one is potential.

§ The case $n = 1$ is trivial since all straight lines are potential curves.

|| Briot et Bouquet, p. 226; Salmon-Fiedler, Höhere Kurven, 2d ed. p. 145.

$$(A\,B\,P_1\,P_2),\quad (A\,B\,P_2\,P_3),\quad \cdots,\quad (A\,B\,P_n\,P_1)$$

are equal, and therefore the set of points $P_1, \cdots, P_n$ is apolar to the pair A, B.

THEOREM XIII. *If a curve is apolar to a point pair, the line through the pair intersects the curve in a set of points apolar to the pair, and the tangents to the curve at these points are concurrent.*

The converse is true only for conics and cubics.

§ 4. *Connection with the Theory of Equations.*

Consider the general equation of the nth degree in one unknown

$$(6)\qquad f(z) \equiv A_0 z^n + A_1 z^{n-1} + \cdots + A_n = 0 \qquad (A_k = b_k + ic_k),$$

with the n roots

$$z_\alpha = x_\alpha + iy_\alpha \qquad (\alpha = 1, 2, \cdots, n).$$

The conjugate equation

$$\bar{f}(z) \equiv \overline{A}_0 z^n + \overline{A}_1 z^{n-1} + \cdots + \overline{A}_n = 0 \qquad (\overline{A}_k = b_k - ic_k),$$

then has the roots

$$(7)\qquad \bar{z}_\alpha = x_\alpha - iy_\alpha \qquad (\alpha = 1, 2, \cdots, n).$$

The complete solution of equation (6) is equivalent directly to the real solution of the system

$$(8)\qquad \varphi(x, y) = 0, \qquad \psi(x, y) = 0,$$

where φ and ψ are the real and imaginary parts of $f(x + iy)$. A problem which then presents itself, namely, the complete solution of this system, is virtually answered in Theorem X: the solutions, by (5), are obtained by solving the linear equations

$$x + iy = z_\alpha, \quad x - iy = \bar{z}_\beta \qquad (\alpha, \beta = 1, 2, \cdots, n).$$

THEOREM XIV. *The complete solution of the auxiliary system* (8) *connected with the equation* (6) *is*

$$x_{\alpha\beta} = \frac{z_\alpha + \bar{z}_\beta}{2}, \quad y_{\alpha\beta} = \frac{z_\alpha - \bar{z}_\beta}{2i} \qquad (\alpha, \beta = 1, 2, \cdots, n).$$

The n real solutions x_α, y_α are obtained by letting $\beta = \alpha$; the remaining solutions may be expressed in terms of these as follows:

$$x_{\alpha\beta} = \frac{x_\alpha + x_\beta}{2} + i\frac{y_\alpha - y_\beta}{2}, \quad y_{\alpha\beta} = \frac{y_\alpha + y_\beta}{2} - i\frac{x_\alpha - x_\beta}{2}.$$

As is well known, the n quantities

$$R(z_\alpha) \qquad (\alpha = 1, 2, \cdots, n),$$

where R denotes any rational function, satisfy an equation of the nth degree

$$F_n(t) = 0,$$

whose coefficients are rational in $A_0, A_1, \cdots, A_n$. This, however, no longer holds when we consider, instead of rational functions of the roots, rational functions of the real and imaginary parts of the roots; but if we consider the n^2 quantities

$$R(x_{\alpha\beta}, y_{\alpha\beta}),$$

they will satisfy an equation of the n^2 degree with coefficients which are rational in terms of the coefficients of φ and ψ, *i. e.*, in terms of $b_0, \cdots, b_n, c_0, \cdots, c_n$. Therefore,

THEOREM XV. *The n quantities*

$$R(x_1, y_1), \quad R(x_2, y_2), \cdots, R(x_n, y_n),$$

are the real roots of an equation of degree n^2 with coefficients which are rational in terms of the real and imaginary parts of the coefficients in (1); *the remaining roots of the equation being*

$$R\left(\frac{x_\alpha + x_\beta}{2} + i\frac{y_\alpha - y_\beta}{2}, \quad \frac{y_\alpha + y_\beta}{2} + i\frac{x_\alpha - x_\beta}{2}\right), \quad (\alpha \neq \beta).$$

This result may easily be extended to functions of any number of roots $R(x_1, y_1, x_2, y_2, \cdots)$, and Theorem XIV may be extended to any system of simultaneous equations.

COLUMBIA UNIVERSITY,
February 25, 1901.

ALTERNATING CURRENT PHENOMENA.

Alternating Current Phenomena. By C. P. STEINMETZ. New York, Office of the Electrical World. Third Edition, 1900. Pp. xx + 525.

To electrical engineers Mr. Steinmetz's book is immediately conspicuous by reason of two distinguishing characteristics:

the first is the employment of a definite mathematical method of presentation, consistently maintained throughout the course of the work, and the second the employment of this method in the analysis of practically every problem in the application of alternating currents of electricity. A glance over the literature of applied electricity reveals no other work which stands forth so prominently in either of these characteristics, and the value of a treatment embracing them both can only be rightly estimated by those who have worked out their basic conceptions of alternating current phenomena and their applications from the conglomerate mass of trigonometry, differential equations, and inaccurate diagrams presented by earlier writers, and, found how inadequate it is for the solution of the problems confronting the engineer of today.

It is the first of these characteristics, namely the method of treatment that is the more interesting to mathematicians, and it is the purpose of this article to review the application of this method; a critical discussion of the complete work from the standpoint of the electrical engineer is not aimed at, and so all reference is omitted to much of the matter that is most valuable but hardly of interest in this place. Briefly stated, the method is the use of the algebra of the complex number in combination with a reference to polar coördinates of the alternating or periodic functions of current and electromotive force. A short consideration of a simple electric circuit carrying an alternating current will facilitate a review of the use to which this method has been put.

Threading or looping with a circuit carrying a current, there is a number of lines of magnetic force due to the passage of the current around the circuit; and this number rises, falls, and reverses with the varying values of the current; the induction or total number of these lines is thus also a periodically varying function "in phase" with the current. Due to this alternating field of force there is induced in the circuit an alternating electromotive force, which is shown by the law of Lenz to have its maximum one-quarter of the time of one complete period later than the inducing field, and so 90° later than that of the current, 360° representing a complete period; this is the counter electromotive force of self-induction. Due to the resistance of the conductor there is a consumption of electromotive force when the current flows, proportional to the current at each instant and so alternating and in phase with the current; this may be considered a counter electromotive force, 180° away from the current. If there is a condenser or electrical

capacity in the circuit there is a third electromotive force which may be shown to have its phase 90° in advance of the current. It is the presence of these several angularly separated electromotive forces which causes the apparent failure of Ohm's law in the case of alternating current circuits. The impressed electromotive force necessary to cause the current to flow must overcome these several electromotive forces, *i. e.*, must have components angularly separated, and so in general will not be in phase with the current. It is to be noted that the difference in phase between current and counter electromotive force is either 0° or plus or minus 90°.

The fundamental principles suggested in the foregoing paragraph are assumed in the opening chapter of Mr. Steinmetz's book, as is also the form of the expression giving the value of the "impedance" or ratio of impressed E.M.F. to current. This expression for the impedance is $z = \sqrt{r^2 + x^2}$, r being the resistance or ratio of the in-phase component of the E.M.F. to the total current, and x the "reactance" or ratio of the out-of-phase component of the E.M.F. to the total current; since as indicated above there are two out-of-phase E.M.F.'s, one 90° in advance of, the other lagging 90° behind the current, x will take its value from the difference between the two E.M.F.'s due to self-induction and capacity, since they differ in phase by 180° or are opposed to each other. The values of x for self-induction and for capacity are calculated in terms of the frequency or number of periods per second and the constants of the circuit.

Passing now to chapter IV, the alternating or sinusoidal wave, represented by time as abscissa and instantaneous value as ordinate, is referred to polar coördinates, giving the circle as the curve; and for each complete period the circle is traversed twice, negative values of the function being taken when a reverse direction must be taken by the radius vector in order to intersect the curve. Thus the intercept on any radius vector gives the instantaneous value of the wave at the time represented by the amplitude of the vector. Since any particular value determines the wave, the step is then made of letting the diameter of the characteristic circle represent the wave, by its length the intensity and by its amplitude the phase; thus on the same diagram any number of E.M.F.'s and currents in a circuit differing in intensity and in phase may be represented by radii vectors of different lengths and amplitudes. The author here tacitly assumes that the current has at any instant

the same value throughout the whole length of the circuit, *i. e.*, that the phase of the current wave does not change from point to point; cases in which this is unwarranted are considered in chapter XIII. The possibility of combining or resolving vectors of the same nature by the parallelogram law is then shown by considering combined instantaneous values on any chosen radius vector; thus the resultant of two electromotive forces. for instance, is represented by the diagonal of the parallelogram formed on the two radii representing their intensities and phases. The graphical method here clearly developed gives perhaps the clearest insight possible into the mutual relations of the several alternating sine waves entering into any problem.

Owing to the widely differing magnitudes of the alternating waves to be represented in the same diagram, the graphical method is not well suited for numerical calculation, and in chapter V the author extends the graphical treatment into the symbolic method, which, instead of denoting the vector representing the sine wave by the polar coördinates of intensity I and amplitude ω, uses the rectangular coördinates $a = I\cos\omega$ and $b = I\sin\omega$, thus avoiding the use of trigonometric functions in the combination or resolution of waves. Extending the relations of the parallelogram law: "Sine waves are combined or resolved by adding or subtracting their rectangular components." To distinguish between the two components, the symbol j is put before the vertical component, $I = a + jb$, meaning that a is the horizontal and b the vertical component of the wave I and that they are combined in the wave of resultant intensity $i = \sqrt{a^2 + b^2}$; similarly $a - jb$ is a wave with a as horizontal and $-b$ as vertical component. The next step brings in the full significance of the method; multiplying the symbolic expression $a + jb$ by -1 evidently gives $-a - jb$, or a wave of equal intensity but differing in phase by 180°; a wave of equal intensity but lagging in phase by 90° (clockwise rotation) is evidently represented by $ja - b$; this expression may be derived by multiplying the expression $a + jb$ by j if upon the "hitherto meaningless symbol j" the condition be imposed $j^2 = -1$; and similarly multiplying by $-j$ advances the wave through 90° or one quarter of a period. The symbol j is thus seen to be the imaginary unit, and the sine wave is represented by the complex quantity of the type $a + jb$; the letter j is used instead of the usual i, since the latter so commonly in electrical literature denotes the current. "As the imaginary unit j has no meaning in the system of ordinary numbers, this defi-

nition of $j = \sqrt{-1}$ does not contradict its original introduction as a meaningless symbol." Thus $a + jb$ means a wave of intensity $i = \sqrt{a^2 + b^2}$ and of phase $\omega = \tan^{-1}\frac{b}{a}$; it may also be represented by $i(\cos\omega + j\sin\omega)$ and also by $ie^{j\omega}$. A further extension of method now gives: "Sine waves may be combined or resolved by adding or subtracting their complex algebraic expressions." The complex expression for the impedance is then developed and may here serve as a simple instance of the use of the method. A current $I = i + ji'$ flows in a circuit of resistance r and reactance x; the E.M.F. consumed by resistance is in phase with the current and is $rI = ri + jri'$; this E.M.F. must be supplied by the impressed E.M.F., as must also be an E.M.F. necessary to overcome the counter E.M.F. due to the reactance x; this E.M.F., if it be due to self induction, lags 90° behind the current and is therefore represented by $ixI = jxi - xi'$ (if x be due to capacity, by $-jxI = xi' - jxi$); the component of the impressed E.M.F. to overcome this is evidently $-jxI = -jxi + xi'$; and the E.M.F. to overcome both resistance r, and reactance x is $E = (r - jx)\,I$; or, the ratio of electromotive force to current, *i. e.*, the impedance, has for its complex expression $Z = r - jx$. The relation $E = ZI$, the complex values of the three quantities being used, may be handled in any of its three forms, giving a simple complex expression for any one of the quantities in terms of the components of the other two, since the imaginary may be easily eliminated from a denominator, and thus the real and imaginary components separated.

From the foregoing, it is seen that the total impedance of a circuit, consisting of any number of portions differing as to r and x, and connected in series, may be obtained by adding the complex expressions for the impedance of the several portions. The total impedance being known, the E.M.F. necessary to supply any given value of the current to such a circuit may at once be had, together with its phase relation to the current. This is the general nature of one class of problems. If, however, the circuit has several branches, or if several currents are supplied by the same E.M.F., the total impedance of the circuit is not a simple expression, just as the resistance of a number of branches connected in parallel is not a simple expression in the resistances of the several branches. In the latter case, however, the joint conducting power, *i. e.*, the reciprocal of the resistance, is the sum of the "conductances" of the several branches. So for parallel connected branches the author derives in chapter VII

a method for combining, by the addition of complex quantities, the effects of the several impedances of a branched circuit. The total current supplied by the impressed E. M.F. is the sum of the currents in the several branches, attention being paid to their phase relations, *i. e.*, the complex expressions of the currents are to be added. To simplify this, Ohm's law, which now holds if complex expressions are used, is put in the form $I = EY$, where Y, being the reciprocal of Z, is a complex quantity, and the values of I for the several branches are now readily added. Y is called the admittance, and the total admittance of a branched circuit is the sum of the complex expressions of the individual admittances. Y, being complex, is of the form $g + jb$. We have

$$Y = g + jb = \frac{1}{Z} = \frac{1}{r - jx} = \frac{r + jx}{r^2 + x^2}.$$

Therefore

$$g = \frac{r}{r^2 + x^2} \quad \text{and} \quad b = \frac{x}{r^2 + x^2},$$

so that the expressions for Y and Z for any branch or combination of branches are readily derived one from the other.

Chapters VIII and IX give a complete investigation of the various types of series and parallel circuits, of the effects upon regulation and phase difference of the relative values of r, x, g, and b; and results heretofore obtained only in very complicated form are reduced to simple algebraic expressions. A single simple case will suffice here, but no electrical engineer should fail thoroughly to digest the contents of these two chapters.

A reactance x_0 is inserted in series with a load circuit of impedance $Z = r - jx$, and an E.M.F. E impressed upon the whole. The total impedance is $Z - jx_0$ or $r - j(x + x_0)$; the current is

$$I = \frac{E_0}{r - j(x + x_0)},$$

with the absolute value $\dfrac{E_0}{\sqrt{r^2 + (x + x_0)^2}} = \dfrac{E_0}{\sqrt{z^2 + 2xx_0 + x_0^2}}$;

the E.M.F. on the receiver or load circuit is

$$E = IZ = \frac{E_0(r - jx)}{r - j(x + x_0)} = E_0 \sqrt{\frac{r^2 + x^2}{r^2 + (x + x_0)^2}}$$

$$= \frac{E_0 z^2}{\sqrt{z^2 + 2xx_0 + x_0^2}}.$$

Generally in such a case the value of E, as compared with E_0, is of prime importance, and the above expression gives means of controlling E by a proper variation of x_0; thus $E = E_0$ if $x_0 = -2x$; if $x_0 < -2x$ it raises, if $x_0 > -2x$ it lowers the voltage; if x and x_0 have the same sign E is always less than E_0. It is to be remembered that a positive value of x is given by self-induction, a negative value by capacity. The difference in phase between current and E.M.F. is gotten, as indicated above, from the expression for the impedance; here the difference in phase in the load circuit is $\omega = \tan^{-1}\frac{x}{r}$; and in the supply or generator circuit $\omega' = \tan^{-1}\frac{x + x_0}{r}$.

Chapter XII is new in the third edition; in it the author attempts to extend the symbolic method to quantities of double frequency, such as the power. At any instant the flow of power in an alternating current circuit is the product of the instantaneous values of current and E.M.F. If two sine waves, *e. g.*, one of current and one of E.M.F. differing in phase, be drawn in rectangular coördinates, and if also a curve representing the product of their instantaneous values be drawn, it is found that while either the current or E.M.F. has passed through half a period, the curve of products, or the power wave, has passed through a complete period; that is, the power has double the frequency of the current and E.M.F., and so may not be represented on the same vector diagram with them. The area of positive values in the power curve represents power given into the medium by the circuit, that of negative values power returned to the circuit from the medium, this power having been stored there in the forms of a magnetic field and an electrostatic strain; the difference is the true expenditure of power. The product of the complex expressions for current and E.M.F., $(e' + je'')(i' + ji'') = (e'i' - e''i'') + j(e''i' + e'i'')$, does not represent the power, since it is an expression of the same frequency as the current and E.M.F.; suppose, however, since the power is of double frequency, the phase angle be doubled in the above expression; *i. e.*, instead of $j^2 = -1$, corresponding to a rotation through 180°, we now have $j^2 = +1$, or 360° rotation, and multiplication by j merely reverses the sign or rotates through 180°. The product then becomes $(e'i' + e''i'') + j(e''i' - e'i'')$, the first term of which is the real power $EI \cos \omega$, and the second the author calls the "wattless power," or $EI \sin \omega$. The chapter is interesting in explaining the apparent failure of the product of the two complex expressions to represent the power,

but is unattractive, except to those versed in non-commutative algebra, because of the necessity of remembering that $j \times 1 = j$ is not the same as $1 \times j = -j$.

Chapter XIII is devoted to those cases where it is not permissible to assume that at any instant the value of the current is the same throughout the circuit; an instance is a submarine cable or any line along which capacity in some quantity is uniformly distributed. In such cases the simple vector diagram and the algebra of complex quantities do not suffice; however, by considering the values of r, x, g, b per unit length of line, the author forms and solves the differential equations for both current and E.M.F., as varying from point to point in the line. While exhaustive in discussion and most useful, the chapter offers no striking application of the symbolic method.

In chapters XIV, XV, and XVI the symbolic method is extended to the analysis of the transformer and the induction motor, *i. e.*, the motor with rotating magnetic field; they are shown to belong to the same general type of apparatus (a fact not before recognized), called by the author the general alternating current transformer. Consider the simple transformer consisting of a magnetic circuit interlinked with two electric circuits, a primary and a secondary. The primary circuit carrying current sets up a field in the magnetic circuit, which induces an E.M.F. in the secondary which supplies current to its load. The secondary is now considered as a simple circuit with a given impressed E.M.F., having an internal impedance $Z_1 = r_1 - jx_1$ due to its resistance and self-induction, and feeding a load circuit of impedance $Z = r - jx$. Since the same magnetic circuit links with both coils, the actions in the secondary are shown to be reducible to the primary by the ratio of the numbers of turns in each; the primary has also its internal impedance $Z_0 = r_0 - jx_0$; combining the primary impedance with the reduced values of the secondary circuit, the effect of the whole transformer is brought to the expression of a single impedance. By proper attention to the difference in frequency between primary and secondary due to the "slip" of the armature (secondary) behind the rotating field due to the primary, the same method of procedure is adopted for the induction motor.

Chapter XXIV is a most interesting extension of the symbolic method to the representation of the general alternating wave as distinguished from the simple sine wave of the type $A = a_0 \cos(\varphi - \alpha)$; to the latter only is the vector representation $A = a' + ja'' = a(\cos\alpha + j\sin\alpha)$ applicable.

If the two half periods of a wave are similar, the even harmonics are absent and the general wave is expressed by

$$A = A_1 \cos(\varphi - \alpha_1) + A_3 \cos(3\varphi - \alpha_3) + A_5 \cos(5\varphi - \alpha_5) + \cdots,$$

which may not be represented by a single complex vector quantity. The individual harmonics, however, of this general wave are independent and no products appear, so that each may be represented by a complex symbol and the symbolic expression for the general wave is

$$A = \sum_1^\infty (2n-1)(a_n' + j_n a_n'');$$

here $j_n = \sqrt{-1}$ always, but the index of j_n denotes that the j's of different indices, while equal algebraically, physically represent different frequencies and so cannot be combined. The general wave of E.M.F. is thus represented by

$$E = \sum_1^\infty (2n-1)(e_n' + j_n e_n''),$$

and the current by a similar expression in the i's. The expression for the impedance undergoes some change; the values of x, the reactance, when due to self-induction, are directly, when due to capacity, inversely proportional to the frequency; there is also a part independent of the frequency; thus the impedance of a circuit will have different values for the several harmonics and its general expression is

$$Z = r - j_n \left(n x_m + x_0 + \frac{x_c}{n} \right).$$

Operations according to Ohm's law may now be performed on the general wave quantities E, I, and Z, just as on the simple sine wave; multiplication and division, however, being only performed on those terms having the same index n.

A common and most useful medium for the use of alternating currents is the so called polyphase system in which several equal E.M.F.'s differing in phase by the same angle are generated in the same machine; the induction motor is the most conspicuous form of apparatus depending on the polyphase system. The symbolic method lends itself admirably to the representation of such a system by means of the nth roots of unity. In the polar diagram the n E.M.F.'s of an n-phase system are represented by n equal vectors

following each other under equal angles $2/\pi n$. In symbolic notation, advance or rotation through an angle $2\pi/n$ is represented by multiplying by the quantity $\cos 2\pi/n + j \sin 2\pi/n$, and so the E.M.F.'s of a polyphase system are

$$E, \quad E\left(\cos\frac{2\pi}{n} + j \sin\frac{2\pi}{n}\right), \quad E\left(\cos\frac{4\pi}{n} + j \sin\frac{2\pi}{n}\right), \text{ etc.}$$

In chapters XXVI and XXVIII the author handles this application for the deduction of the expression for the rotating magnetic field, the ring and star E.M.F.'s of interlinked systems, and other matters of general use, but up to this time wanting an analytical expression.

The writer has only gratitude to express at the appearance of this work, and his one regret is that its author did not also include in it his recent articles on the rotary converter.

JOHN B. WHITEHEAD, JR.

JOHNS HOPKINS UNIVERSITY,
April 18, 1901.

SHORTER NOTICE.

Leçons Nouvelles sur les Applications Géométriques du Calcul Différentiel. By W. DE TANNENBERG. Paris, A. Hermann, 1899. 192 pp.

THIS volume, which M. de Tannenberg has contributed to the literature of the theory of curves and surfaces, is very opportune. We have wanted a book which would make possible for the beginner a knowledge of the more fundamental geometrical applications of the calculus and in a way which would prepare him for the treatises of Darboux and Bianchi. To be sure, this field has been covered, more or less, in the chapters devoted to geometrical applications in the French treatises on analysis—notably by Jordan, Picard, Appell—but rather as examples of the methods of analysis and not standing forth as a systematic development of the elements of another field of mathematics. Again, there have been in recent years, quite a number of shorter treatises with just the scope of the volume under discussion, but their treatment of the subject has been along lines quite different from the well known methods of the calculus: Ricci in his Lezioni arrives at the results by the study of

differential forms, Méray makes use of the methods which he has developed in his Leçons nouvelles sur l'analyse infinitésimale, Fehr adopts the vector method of Grassman.

But M. de Tannenberg has given us a treatment of the subject by methods which involve only the simplest notions of the calculus, and in a way that is at the same time clear, concise, and rigorous. Limiting himself to the study of curves and surfaces in the region of ordinary points, he considers the cartesian coördinates as developed according to ascending powers of one or two variables, and from the study of these developments determines many of the geometrical properties of the given locus. As one reads, he is impressed with the simplicity of the methods; very often an important result follows as a corollary, or is embodied in a remark, and at times the examples which are scattered through the book serve the double purpose of illustration and a step toward the further development of the subject. Very little is said about plane curves; and, when treated, it is simply as a particular case of curves in general.

The book has two main divisions: the first, consisting of two parts, treats of the descriptive properties of curves and surfaces; the second, in three parts, treats of their metric properties.

Of particular interest in the first division is the treatment of the theory of envelopes, whether of curves or surfaces. The conditions necessary and sufficient for the existence of envelopes are set forth so clearly that the beginner should not experience the usual difficulty with this subject. The second part closes with a brief discussion of systems of lines —ruled surfaces, congruences, and linear complexes.

In the third part of the book and the first of the second division, the notion of the geometrical derivative of a segment is introduced and is of great service in the definition of first curvature and torsion of a curve, and the derivation of the Serret formulæ. As an application of these formulas, the variation of a segment of a right line is discussed, and the results thus obtained enable the author to simplify the proofs of many theorems in the theory of surfaces. In particular, they are applied to the study of skew and developable surfaces; this forms the fourth part.

The fifth part is the most important; it comprises nearly one-half of the book and is devoted to the theory of curved surfaces. First the six fundamental functions, characteristic of a surface, are defined and the relations which they satisfy are found; then the general properties of curves traced on surfaces are discussed. From the theorems of

Meusnier and Bonnet follow the definitions of asymptotic lines and lines of curvature; their properties, together with those of conjugate lines, are then discussed and geodesic lines are defined. After an application of the preceding theories to several of the simpler surfaces—cones, surfaces of revolution, quadrics, developables, surfaces of Monge—the book closes with a treatment of the general equation of geodesic lines.

The print is good, the figures are helpful and the general arrangement is very attractive.

L. P. EISENHART.

NOTES.

THE secretaryship of Section A of the American association for the advancement of science has been filled by the appointment of Dr. G. A. MILLER. Papers intended for the programme of the Denver meeting (August 24–31) of Section A, should be sent to Dr. Miller, at 115 Cook St., Ithaca, N. Y.

THE April number of the *American Journal of Mathematics* (volume 23, number 2) contains the following articles: "The cross-ratio group of 120 quadratic Cremona transformations of the plane," by H. E. SLAUGHT; "Memoir on the algebra of symbolic logic," by A. N. WHITEHEAD; "On a special form of annular surfaces," by V. SNYDER; "On the transitive substitution groups whose order is a power of a prime number," by G. A. MILLER; "Geometry on the cubic scroll of the second kind," by F. C. FERRY.

AT the regular meeting of the London mathematical society held on May 9, 1901, the following papers were read: "A case of algebraic partitionment," by Major P. A. MACMAHON; "On the series whose terms are the cubes and higher powers of the binomial coefficients," by Major P. A. MACMAHON; "A property of recurring series," by Dr. G. B. MATHEWS; "The product of two spherical surface harmonic functions," by Mr. J. B. DALE.

THE German Society for the promotion of instruction in mathematics and science held its tenth annual meeting the last week of May at Giessen under the presidency of Dr. Rausch. The society has nine hundred members.

THE committee elected to judge the competition in 1901 for the Francoeur and Poncelet prizes of the Paris academy of sciences consists of Professors C. JORDAN, H. POINCARÉ, E. PICARD, P. APPELL, and MAURICE LEVY.

THE following additional volumes (see BULLETIN, volume 6, pp. 306 and 355) are announced as either in preparation or under the press for B. G. Teubner's series of text-books in the mathematical sciences: G. BOHLMANN, The mathematics of insurance; G. H. BRYAN, Thermodynamics; L. E. DICKSON, Linear groups, with an exposition of the Galois field, theory (English, in press); G. ENESTRÖM, with the coöperation of other scholars, Handbook on the history of mathematics; PH. FURTWÄNGLER, Mechanics of the simplest kinds of physical apparatus and of experiments; M. GRÜBLER, Hydraulic motors; L. HENNEBERG, Text-book of graphical statics; K. HEUN, The kinetic problems of the modern theory of machines; G. JUNG, Geometry of masses; H. LAMB, Acoustics; A. LOEWY, Lectures on the theory of linear substitution groups; A. E. H. LOVE, Textbook of hydrodynamics; A. E. H. LOVE, Theory of elasticity; W. MEYERHOFFER, The mathematical foundations of chemistry; E. OVAZZA (from the field of mechanics); FR. POCKELS, Optics of crystals; P. STÄCKEL, Textbook of general dynamics; H. E. TIMERDING, Theory of systems of vectors and screws; A. Voss, Principles of rational mechanics.

Of the volumes previously announced three have appeared, viz., E. PASCAL, Determinants; O. STOLZ and J. A. GMEINER, Theoretical arithmetic, Part I; E. v. WEBER, Lectures on the problem of Pfaff. Three others are in press, viz., L. E. DICKSON, Linear groups (see above); G. LORIA, The theory and history of special algebraic and transcendental plane curves; E. NETTO, Theory of combinations.

SUBSCRIPTIONS are invited toward the publication of the complete works of the late Professor E. BELTRAMI (see BULLETIN, volume 7, p. 46), which has been undertaken by the faculty of science of the University of Rome. There will be four large volumes with about 2000 pages in all. A copy of the complete works will be sent to those who subscribe not less than fifty lire. Subscriptions may be sent to ISAIA SONZOGNO, Piazza S. Pietro in Vincoli, 5, Rome.

IT appears, from *Nature*, that the English civil service commission no longer insists, in its examinations in geometry, on preserving the Euclidean sequence of propositions, but now accepts correct demonstrations whether those of Euclid or not. Recent papers also encourage teaching of a less abstract character than that usually associated with Euclid's geometry.

To the list of mathematical courses to be given at Colum-

bia University in 1901–1902, as announced in the last number of the BULLETIN, there is to be added a course on "The history of mathematics," three hours weekly, by Professor D. E. SMITH.

A BUST of GAUSS has been placed in one of the halls of the University of Berlin.

PROFESSOR E. H. MOORE is one of the five new members elected to the National academy of sciences at its annual meeting April 16th–18th.

DR. J. A. GMEINER has been made professor extraordinarius of mathematics at the German University of Prague.

PROFESSOR GINO FANO, now at Messina, has been made assistant professor of projective and descriptive geometry at the University of Turin. Professor U. DINI of the University of Pisa has been elected a member of the superior council of instruction of Italy.

AT Yale University Professor J. E. CLARK, James E. English professor of mathematics in the Sheffield Scientific School since 1873, has resigned because of poor health. He has been made professor emeritus, and Professor P. F. SMITH has received the title of James E. English professor. Dr. E. R. HEDRICK has been appointed instructor in mathematics in the Sheffield Scientific School.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BALL (W. W. R.). A short account of the history of mathematics. 3d edition. London and New York, Macmillan, 1901. 12mo. 552 pp. Cloth. 10s.

BEMAN (W. W.). See DEDEKIND (R.).

BLICHFELDT (H.). On a certain class of groups of transformations in space of three dimensions. (Diss.) Leipzig, 1900. 8vo. 61 pp.

BONNEL (J.). Note sur les systèmes de géométrie et l'atome. Paris, Gauthier-Villars, 1900. 8vo. 15 pp. Fr. 1.00

CADENAT (A.). See RUSSELL (B. A. W.).

CARNOY (J.). Cours d'algèbre supérieure: principes de la théorie des déterminants; théorie des équations; introduction à la théorie des formes algébriques. 2e édition. Paris, Gauthier-Villars, 1900. 8vo. Fr. 11.00

COUTURAT (L.). See RUSSELL (B. A. W.).

CRIVETZ (T.). Essai sur l'équidistante. Bucarest, Göbl, 1900. 8vo. 50 pp.

DARREYE (A.). Polare Felder und Kegelschnitte mit gemeinsamem Poldreieck. (Diss.) Strassburg, 1900. 8vo. 41 pp., 2 plates.

DAVISSON (S. C.). Ueber die geodätische Linie der Manigfaltigkeit $ds^2 = dx^2 + \sin^2 x\, dy + dz^2$. (Diss.) Tübingen, 1900. 8vo. 22 pp.

DEDEKIND (R.). Essays on the theory of numbers: (1) Continuity and irrational numbers; (2) The nature and meaning of numbers. Authorized translation by W. W. Beman. Chicago, Open Court Publishing Company, 1901. 12mo. 4 + 115 pp. Cloth. $0.75

DÜRLL (W.). Die Probleme des logarithmischen Potentials für eine von zwei Kreisbogen begrenzte ebene Fläche. (Diss.) Leipzig [1900]. 8vo. 134 pp., 2 plates.

ERMAKOF (V. P.). Lectures on analytic geometry delivered in the University and in the Polytechnic Institute. Part II: Solid geometry. Kief, 1900. 8vo. 208 pp. (Russian.) M. 7.00

——. Lectures on the integral calculus. Parts I and II. Kief, 1900. 8vo. 350 pp. (Russian.) M. 9.00

GUTZMER (A.), HENSEL (K.), HEUN (K.). See JAHRESBERICHT.

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